INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 40, Northern Autumn 2018 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Is it possible to place a line segment XY inside a regular pentagon $A_1A_2A_3A_4A_5$ so that all five angles $\angle XA_iY$ $(i = 1, \ldots, 5)$ are equal? (3 points)

2. Determine all positive integers n such that the numbers $1, 2, \ldots, 2n$ can be divided into pairs so that the product of sums of the numbers in each pair is a perfect square. (4 points)

3. In parallelogram $ABCD$, $\angle A$ is acute. A point N is chosen on the side AB so that $CN = AB$. Suppose that the line AD is tangent to the circumcircle of triangle CBN. Prove that D is the point of tangency. (5 points)

4. A nine-digit integer is called *beautiful* if all of its digits are different. Prove that there exist at least 1000 beautiful numbers, each of which is divisible by 37. (5 points)

5. Petya is placing 500 kings on a 100×50 chess board so that the kings don't attack one another. Vasya is placing 500 kings on white squares of a 100×100 chess board so that the kings don't attack one another. Who has more ways to place the kings? (5 points)

O Level Senior Paper Solutions

Prepared by Oleksiy Yevdokimov and Greg Gamble

1. No, it is not possible. Assume the contrary, i.e. it is possible to place a line segment XY inside a regular pentagon $A_1A_2A_3A_4A_5$ so that all five angles $\angle XA_iY$ ($i = 1, \ldots, 5$) are equal. Then the line XY cannot pass through any vertex A_i, else such an angle $\angle XA_iY$ would be equal to 0 which is not possible if all five angles $\angle XA_iY$ are equal, as per assumption. Thus, by the Pigeon-Hole Principle three vertices of the pentagon, say A_1, A_2 and A_3 are on the same side of the line XY. Consequently, points A_1, A_2, A_3, X and Y lie on the same circle, since

$$\angle XA_1Y = \angle XA_2Y = \angle XA_3Y$$

and these angles are subtended by the same chord XY. However, this circle must coincide with the circumcircle of the pentagon $A_1A_2A_3A_4A_5$. Since the points X and Y are distinct from the vertices of the pentagon, both points must lie outside the pentagon, and so we come to a contradiction. Hence, our assumption is false.

2. Let $\mathcal{S}$ be the chosen partition of $\{1, 2, \ldots, 2n\}$ into pairs, and P be its corresponding product of pair-sums.

Solution 1. There is a partition $\mathcal{S}$ for which P is a perfect square, for any $n > 1$.

For $n = 1$, there is only one choice for $\mathcal{S}$, namely $\{\{1, 2\}\}$, and hence necessarily $P = 1 + 2 = 3$, which is not a perfect square.

For even n, we have $n = 2k$ for some integer $k \geq 1$, and we can choose

$$\begin{aligned}&\mathcal{S} = \{\{1, 2n\},\ \{2, 2n-1\}, \ldots, \{n, n+1\}\},\ \text{giving}\\ &P = (1 + 2n)(2 + (2n-1)) \cdots (n + (n+1)) = (2n+1)^n = ((2n+1)^k)^2,\end{aligned}$$

a perfect square.

For $n = 3$, we can choose $\mathcal{S} = \{\{1, 5\}\,\{2, 4\},\ \{3, 6\}\}$, giving $P = 6 \cdot 6 \cdot 9 = 6^2 \cdot 3^2$, a perfect square.

For odd $n > 3$, we have $n = 2k + 1$ for some integer $k > 1$. The key idea is that $n - 3$ is even, so that after partitioning $\{1, 2, \ldots, 6\}$ into pairs as we did for $n = 3$, the remaining $n - 3$ numbers can be partitioned into pairs in the way we did for even n, i.e. we can choose $\mathcal{S}$ and hence P as

$$\begin{aligned}\mathcal{S} &= \{\{1, 5\},\ \{2, 4\},\ \{3, 6\}\,\{7, 2n\},\ \{8, 2n-1\}\,\{n+3, n+4\}\},\\ P &= 6 \cdot 6 \cdot 9 \cdot (7 + 2n)(8 + (2n-1)) \cdots ((n+3) + (n+4))\\ &= 6^2 \cdot 3^2 \cdot (2n+7)^{n-3} = (18(2n+7)^{k-1})^2,\end{aligned}$$

where P is again a perfect square.

Solution 2. There is a partition $\mathcal{S}$ for which P is a perfect square, for any $n > 1$.

The key idea is that any four consecutive integers $a, a+1, a+2, a+3$ can be partitioned into the pairs $\{a, a+3\}$, $\{a+1, a+2\}$ whose contribution to P is a square, namely

$$(a + (a+3))((a+1) + (a+2)) = (2a+3)^2.$$

Thus, if $2n$ is divisible by 4, i.e. n is even, we first partition $\{1, 2, \ldots, 2n\}$ into consecutive sets of 4, and then partition the sets of four as pairs, leading to P as a product of squares.

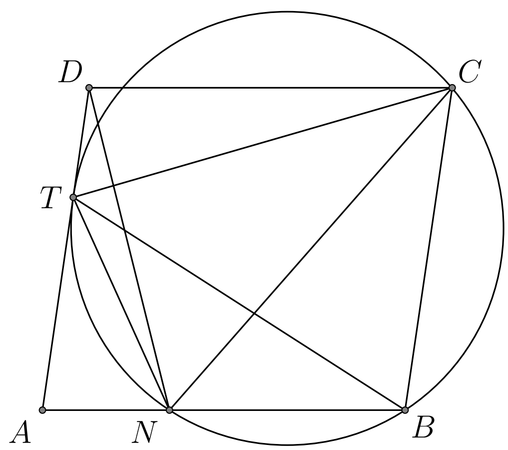
If $2n = 2$, i.e. $n = 1$, as we saw in Solution 1., $P = 3$ is unique and not a square.

This leaves $2n$ is not divisible by 4 having remainder 2, for which $2n \geq 6$. After setting aside the first six numbers, what remains can first be partitioned into consecutive sets of 4. The first 6 can be partitioned into pairs as per Solution 1., giving a contribution of 18^2 to P, and by the strategy above each set of 4 can be partitioned into pairs that give a contribution to P that is a square.

Thus, for all $n > 1$, there is a partition $\mathcal{S}$ for which P is a product of squares, so that as a consequence P is a perfect square.

Note. For $n = 2$ and $n = 3$ the partitions $\mathcal{S}$ that give a square P are unique. For any other n they are not unique.

3. **Solution 1.** Let the point of tangency be T. Our strategy is to show that points T and D coincide.



Since BC and AD are parallel (opposite sides of parallelogram $ABCD$), we have,

$$\begin{aligned} \angle CBT &= \angle ATB, \text{ (alternate angles)} \\ &= \angle BCT, \text{ (by Tangent-Chord Theorem).} \end{aligned}$$

Hence, triangle BTC is isosceles. So,

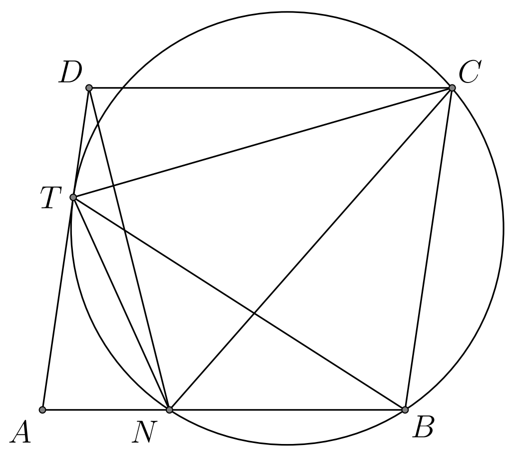
$$\begin{aligned} BT &= CT \\ \angle ABT &= \angle NBT, \text{ (same angle)} \\ &= \angle NCT, \text{ (subtended by same arc)} \\ AB &= CN, \text{ (given).} \end{aligned}$$

So, triangles ABT and NCT are congruent (by the SAS Rule). Therefore,

$$\begin{aligned} \angle BAT &= \angle CNT \\ &= \angle CBT, \text{ (subtended by same arc)} \\ &= \angle BCT, \text{ (by above)} \\ \text{Hence, } \angle ABC + \angle BCT &= \angle ABC + \angle BAT \\ &= 180^\circ, \text{ (cointerior angles).} \end{aligned}$$

Thus, AB and CT are parallel, and so $ABCT$ is also a parallelogram. Hence, points T and D coincide, and hence D is indeed the point of tangency.

Solution 2. Again, let the point of tangency be T. As in Solution 1., we will show that points T and D coincide.



We have,

$$\begin{aligned} CN &= AB, \text{ (given)} \\ &= CD, \text{ (since } ABCD \text{ is a parallelogram).} \end{aligned}$$

So triangle NCD is isosceles. Therefore,

$$\begin{aligned} \angle DNC &= \angle NDC \\ &= \angle AND, \text{ (alternate angles).} \end{aligned}$$

Thus, ND is angle bisector of $\angle ANC$. On the other hand,

$$\begin{aligned} \angle ATN &= \angle TCN, \text{ (by the Tangent-Chord Theorem)} \\ \angle TAN &= 180^\circ - \angle CBN, \text{ (cointerior angles)} \\ &= \angle CTN, \text{ (} BCTN \text{ is cyclic).} \end{aligned}$$

Thus, triangles ATN and TCN are similar. So $\angle ANT = \angle TNC$, and hence NT is also an angle bisector of $\angle ANC$. Thus, the lines NT and ND coincide, and consequently points D and T also coincide. Hence, D is indeed the point of tangency.

4. **Solution 1.** Any nine-digit integer N can be represented in the following way

$$N = 10^6 A + 10^3 B + C = 999 \cdot (1001A + B) + (A + B + C),$$

where A, B and C are the numbers formed by the first three digits of N, the middle three digits of N, and the last three digits of N, respectively.

Since $1 + 2 + \cdots + 9 = 45$, one can partition the digits 1, 2, ..., 9 into three triples, having a common sum of 15; for example, (1, 5, 9), (2, 6, 7) and (3, 4, 8). If the three digits from one triple are placed in the leftmost positions of the numbers A, B and C, the digits of another triple are placed in the middle positions of A, B

and C, and the digits of the third triple are placed in the rightmost positions of A, B and C, then $A+B+C = 15 \cdot 111 = 45 \cdot 37$. Since 37 also divides 999, a beautiful number with such a configuration for A, B and C will be divisible by 37. Since we have six ways to arrange the digits of a triple in the designated position of A, B and C, for each of three triples, and also there are six ways to arrange the triples among the digit positions, we have at least $6^4 = 1296$ beautiful numbers, each of which is divisible by 37.

Solution 2, by William Steinberg. Observe that $3 \cdot 37 = 11$ and $3^3 \cdot 37 = 999$, so that

$$10^2 \equiv -11 \pmod{37} \qquad \text{and} \qquad 10^3 \equiv 1 \pmod{37}.$$

Let the decimal representation of a beautiful number be $\overline{a_8a_7\ldots a_1a_0}$. Then

$$\begin{aligned}\overline{a_8a_7\ldots a_1a_0} &= \sum_{k=0}^{8} a_k \times 10^k \\ &\equiv (a_0+a_3+a_6)\cdot 1 \\ &\quad + (a_1+a_4+a_7)\cdot 10 \\ &\quad + (a_2+a_5+a_8)\cdot(-11) \pmod{37}\end{aligned}$$

Observe that

$$1 + 10 + (-10) \equiv 0 \pmod{37}.$$

So, if it is possible to have

$$\begin{aligned}a_0+a_3+a_6 = a_1+a_4+a_7 &= a_2+a_5+a_8 \\ &= S, \text{ say,}\end{aligned}$$

then $\overline{a_8a_7\ldots a_1a_0} \equiv 0 \pmod{37}$.

Since $1+2+\cdots+9 = 45$, indeed we can have $S = 15$. Let

$$\begin{aligned}\mathcal{D} &= \{\{a_0,a_3,a_6\},\ \{a_1,a_4,a_7\},\ \{a_2,a_5,a_8\}\} \\ \mathcal{T} &= \{\{1,6,8\},\ \{2,4,9\},\ \{3,5,7\}\}\end{aligned}$$

where $\mathcal{D}$ are the digit triples whose sum we want to be 15, and $\mathcal{T}$ is a partition of $\{1,2,\ldots,9\}$ into triples whose sum is 15. Then the digit triples in $\mathcal{T}$ can be assigned to the three triples in $\mathcal{D}$ in $3! = 6$ ways, and after such an assignment the digits of each $\mathcal{T}$-triple can be assigned to the digits of a $\mathcal{D}$-triple in $3! = 6$ ways. So there are at least $6^4 = 1296 > 1000$ beautiful numbers that are divisible by 37.

Solution 3. We consider beautiful numbers that are divisible by 999, observing, since 37 divides 999, that such beautiful numbers are divisible by 37. Let the decimal representation of a beautiful number be $\overline{a_8a_7\ldots a_1a_0}$. Then, noting that $10^3 \equiv 1 \pmod{999}$, we have

$$\begin{aligned}\overline{a_8a_7\ldots a_1a_0} &= \sum_{k=0}^{8} a_k \times 10^k \\ &\equiv (a_0+a_3+a_6)\cdot 1 \\ &\quad + (a_1+a_4+a_7)\cdot 10 \\ &\quad + (a_2+a_5+a_8)\cdot 100 \pmod{999} \\ &\equiv 100x + 10y + z \pmod{999},\end{aligned}$$

where $x = a_2 + a_5 + a_8$, $y = a_1 + a_4 + a_7$, $z = a_0 + a_3 + a_6$.

We claim that if d is a divisor of 999 and $100x + 10y + z$ is divisible by d, then numbers $100y+10z+x$ and $100z+10x+y$ are also divisible by d. (In fact, we only use this property with $d = 999$.) Indeed, this follows from the observation that

$$100y + 10z + x = 10(100x + 10y + z) - 999x,$$

and its companion derived by rotating x to y, y to z, and z to x.

Note that $100 \cdot 8 + 10 \cdot 18 + 19 = 999$. Below, we find five partitions of the non-zero digits into triples whose sums are 8, 18 and 19, respectively:

(1, 3, 4), (5, 6, 7) and (2, 8, 9);
(1, 3, 4), (2, 7, 9) and (5, 6, 8);
(1, 2, 5), (4, 6, 8) and (3, 7, 9);
(1, 2, 5), (3, 7, 8) and (4, 6, 9);
(1, 2, 5), (3, 6, 9) and (4, 7, 8).

So, we have (x, y, z) is a rotation of $(8, 18, 19)$, of which there are 3, and each triple of digits summing to x, y or z can be arranged in 6 ways, and this can be done with each of the 5 partitions of the non-zero digits given above. Hence, there are at least $5 \cdot 3 \cdot 6^3 = 3240$ beautiful numbers divisible by 999 (and hence divisible by 37).

Remark. The intended lower bound for the number of beautiful numbers divisible by 37 for the Senior problem, was 2018. There are many ways of partitioning $\{1, 2, \ldots, 9\}$ into triples whose sum is 15 (so-called *magic square* triples). Solutions 1. and 2. show two ways. If these two different partitions are used in each of Solutions 1. and 2., the lower bound of 1296 beautiful numbers divisible by 37, can be doubled to $2592 > 2018$. A computer search shows that the number of beautiful numbers that are divisible by 37, is in fact 89 712; this compares with $\lfloor 9 \cdot 9!/37 \rfloor = 88\,268$, if beautiful numbers were roughly distributed equally across the residue classes modulo 37.

5. In our discussions below we understand that an $m \times n$ grid has m rows and n columns.

Solution 1. Vasya has more ways to place the kings. We will establish a one-to-one correspondence between all of Petya's ways and some of Vasya's ways. In this way, it will be sufficient to show that Vasya has an extra way of placing the kings that is not part of the established one-to-one correspondence.

Indeed, extend Petya's 100×50 chess board to the right to reach the size 100×100 Vasya has, and reflect all kings placed on black squares in any of Petya's ways with respect to the rightmost border of Petya's initial chess board of size 100×50. We claim, the result of each such reflection is one of Vasya's ways of placing 500 kings on white squares of a 100×100 chess board.

In order to see this, fix one of Petya's king configurations. Since only the black kings have been reflected over the rightmost border of Petya's 100×50 chess board, all kings placed on white squares, in the chosen one of Petya's configurations, stay where they were before the reflection, and they certainly don't attack one another.

Likewise, since the relected kings didn't attack one another on the left half of the 100×100 chess board (which is the chess board of size 100×50) before the reflection, they don't attack one another, after the reflection. Moreover, the square colours of the reflected board must be flipped (black goes to white and white goes to black), to ensure the colours alternate across the axis of relection. So the relected kings end up on white squares. We also note that kings that were on black squares in column 50 have moved to adjacent white squares in column 51 and cannot attack any kings in column 50. Thus all kings in the resulting configuration are on white squares and don't attack one another, and so we have one of Vasya's configurations.

Hence Vasya has at least as many ways as Petya to place the 500 kings.

Now we show there is an extra configuration Vasya has that cannot have come from one of Petya's ways via the above one-to-one correspondence. Choose all the white squares of 50 non-adjacent rows of the 100×100 chess board to place the kings. Then the kings are not attacking for Vasya, but if the kings are reflected back to the left they end up on black squares adjacent (and hence attacking) kings on white squares. So this valid configuration for Vasya is not one of the ones in one-to-one correspondence with one of Petya's.

Thus Vasya has at least one more way to place 500 kings on his board than Petya has on his.

Solution 2. Vasya has more ways to place the kings. As in Solution 1., our strategy is to first establish a one-to-one correspondence between all of Petya's ways and some of Vasya's ways, and then show that Vasya has an extra way of placing the kings that is not part of the established one-to-one correspondence.

This time we start by covering each row of the 100×100 chess board with fifty 1×2 dominos. Now apply a horizontal contraction to the board to get Petya's 100×50 chess board, with the result that each domino shrinks to be 1×1. No matter how Petya places 500 kings (not more than one king to a shrunken domino), Vasya can move each king to the white square on his un-shrunken domino. If two kings somewhere on Vasya's chess board in such a constructed configuration, were to attack one another, then they they would be in adjacent dominos, and so they would be attacking one another in Petya's corresponding configuration. So, in fact, this construction yields a valid one-to-one correspondence between all of Petya's ways and some of Vasya's ways.

Finally, the example of Solution 1., namely that the kings are placed on all the white squares of 50 rows, no two of which are adjacent, is a valid way for Vasya to place the kings that is outside the one-to-one correspondence.

Thus Vasya has at least one more way to place 500 kings on his board than Petya has on his.